# Effects of cross correlation on the relaxation time of a bistable system driven by cross-correlated noise

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We study the effects of correlations between additive and multiplicative noise on relaxation time in a bistable system driven by cross-correlated noise. Using the projection-operator method, we derived an analytic expression for the relaxation time $T_c$ of the system, which is the function of additive ($\alpha$) and multiplicative ($D$) noise intensities, correlation intensity $\lambda$ of noise, and correlation time $\tau$ of noise. After introducing a noise intensity ratio and a dimensionless parameter $R=D/\alpha$, and then performing numerical computations, we find the following: (i) For the case of $R<1$, the relaxation time $T_c$ increases as $R$ increases. (ii) For the cases of $R\geqslant 1$, there is a one-peak structure on the $T_c$-$R$ plot and the effects of cross-correlated noise on the relaxation time are very notable. (iii) For the case of $R<1$, $T_c$ almost does not change with both $\lambda$ and $\tau$, and for the cases of $R\geqslant 1$, $T_c$ decreases as $\lambda$ increases, however $T_c$ increases as $\tau$ increases. $\lambda$ and $\tau$ play opposite roles in $T_c$, i.e., $\lambda$ enhances the fluctuation decay of dynamical variable and $\tau$ slows down the fluctuation decay of dynamical variable.

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## I. INTRODUCTION

In most of the previous works, noise forces that are present simultaneously in the stochastic systems were usually treated as random variables uncorrelated with each other. However, noises in some stochastic processes may have a common origin and thus can be correlated [1,2]. Also Madureira *et al.* pointed out that fluctuations in some of the model parameters lead to noise contributions of both additive and multiplicative character, and are also not independent [3].

The effects of correlations between additive and multiplicative noise, either on a stationary state or on dynamics of the bistable potential system, have been widely studied [2–12]. It is proved that the transform of the stationary probability density from unimodal to bimodal is strongly influenced by both correlation intensity and correlation time of cross-correlated noise, and this can induce reentrant phase transition [4,6–8]. The presence of correlations between additive and multiplicative noise can bring about a giant suppression of the active rate [3,9]. The fluctuation-induced transport exists as a noise correlation effect [10]. The correlation strength of cross-correlated noises plays very important roles in the mean first-passage time of the bistable system driven by cross-correlated noises [11]; so does the correlation time between additive and multiplicative noise [12].

The correlation functions and the associated relaxation time are used as a dynamic characterization of the steady state fluctuations in nonequilibrium system [13]. Early investigations of the characteristic behavior of the relaxation time were limited to the case of uncorrelated noise, for example, the relaxation time was calculated both for the white and nonwhite noise cases by means of a numerical simulation of stochastic differential equation [14] and by a projector-operator technique [15]. The study of the relaxation time of the system under the influence of cross-correlated noise has not been reported. Because the presence of cross-correlated noise changes the dynamics of the system [2–12], it is predicted that there may exist some new cross-correlated effects on the relaxation time of the system.

The objective of this paper is to discuss the effects of correlations between additive and multiplicative noises on the relaxation time of the bistable system. In Sec. II, the properties of stationary probability distribution of the system was discussed and the analytic expression of the relaxation time for a bistable system coupled to correlated noises was derived by means of the projection-operator method. In Sec. III, a brief conclusion is given.

## II. THE STATIONARY PROBABILITY DISTRIBUTION AND THE RELAXATION TIME OF THE BISTABLE SYSTEM

To calculate the relaxation time of a bistable system by means of the projection-operator method, we need the stationary probability distribution (SPD) of the system. First, we derive SPD of the system and then calculate the relaxation time of the system.

### A. The stationary probability distribution of the bistable system

Consider a conventional symmetric bistable system driven by cross-correlated noise. Its Langevin equation reads

$$\frac{dx}{dt}=x-x^3+x\xi(t)+\eta(t), \tag{1}$$

where $\xi(t)$ and $\eta(t)$ are Gaussian white noise with zero mean, and

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

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$$\langle \eta(t)\eta(t')\rangle = 2\alpha\delta(t-t'). \tag{3}$$

Here $\alpha$ and $D$ are the additive and multiplicative noise intensities, respectively. The correlation times of the correlations between $\xi(t)$ and $\eta(t)$ are nonzero [6,11,12]. Here, assume

$$\langle \xi(t)\eta(t')\rangle = \langle \eta(t)\xi(t')\rangle = \frac{\lambda\sqrt{\alpha D}}{\tau}\exp[-|t-t'|/\tau]$$
$$\rightarrow 2\lambda\sqrt{\alpha D}\,\delta(t-t') \quad \text{as} \quad \tau\rightarrow 0, \tag{4}$$

where $\tau$ is the correlation time of the correlations between $\xi(t)$ and $\eta(t)$, and $\lambda$ denotes the strength of correlation between $\eta(t)$ and $\xi(t)$. The potential

$$V(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 \tag{5}$$

corresponding to Eq. (1) has two stable states $x_1=-1$, $x_2=1$, and an unstable state $x_0=0$. By virtue of the Novikov theorem [16], Fox's approach [17], and ansatz of Hanggi *et al.* [18], the approximate Fokker-Planck equation corresponding to Eq. (1) with Eqs. (2)–(4) is obtained [12]:

$$\frac{\partial P(x,t)}{\partial t} = L_{\mathrm{FP}}P(x,t), \tag{6}$$

$$L_{\mathrm{FP}} = -\frac{\partial}{\partial x}f(x) + \frac{\partial^2}{\partial x^2}G(x), \tag{7}$$

where

$$f(x) = x - x^3 + Dx + \frac{2\lambda\sqrt{\alpha D}}{1+2\tau} \tag{8}$$

and

$$G(x) = Dx^2 + \frac{2\lambda\sqrt{\alpha D}}{1+2\tau}x + \alpha. \tag{9}$$

Note that this approximate Fokker-Planck equation holds under the condition $1+2\tau>0$ for all $\tau$ [6]. Thus, there is no restriction on $\tau$ so that there is not any restriction on all the parameters treated in this case. The stationary probability distribution of the system can be obtained from Eq. (6) with Eqs. (7)–(9)

$$P_{\mathrm{st}}(x) = NG(x)^{-1/2}\exp[-U(x)/\alpha] \quad \text{for} \quad |\lambda|\leqslant 1. \tag{10}$$

Here the generalized potential $U(x)$ is

$$U(x) = -\int^x \frac{y-y^3}{Ry^2 + \frac{2\lambda}{1+2\tau}\sqrt{R}y + 1}\,dy$$
$$= \frac{\beta_1}{\sqrt{\left[1-\left(\frac{\lambda}{1+2\tau}\right)^2\right]R}}\arctan\frac{\sqrt{R}x + \frac{\lambda}{1+2\tau}}{\sqrt{\left[1-\left(\frac{\lambda}{1+2\tau}\right)^2\right]}}$$
$$-\frac{2\lambda}{(1+2\tau)R\sqrt{R}}x + \frac{x^2}{2R} + \beta_2\ln G(x), \tag{11}$$

where

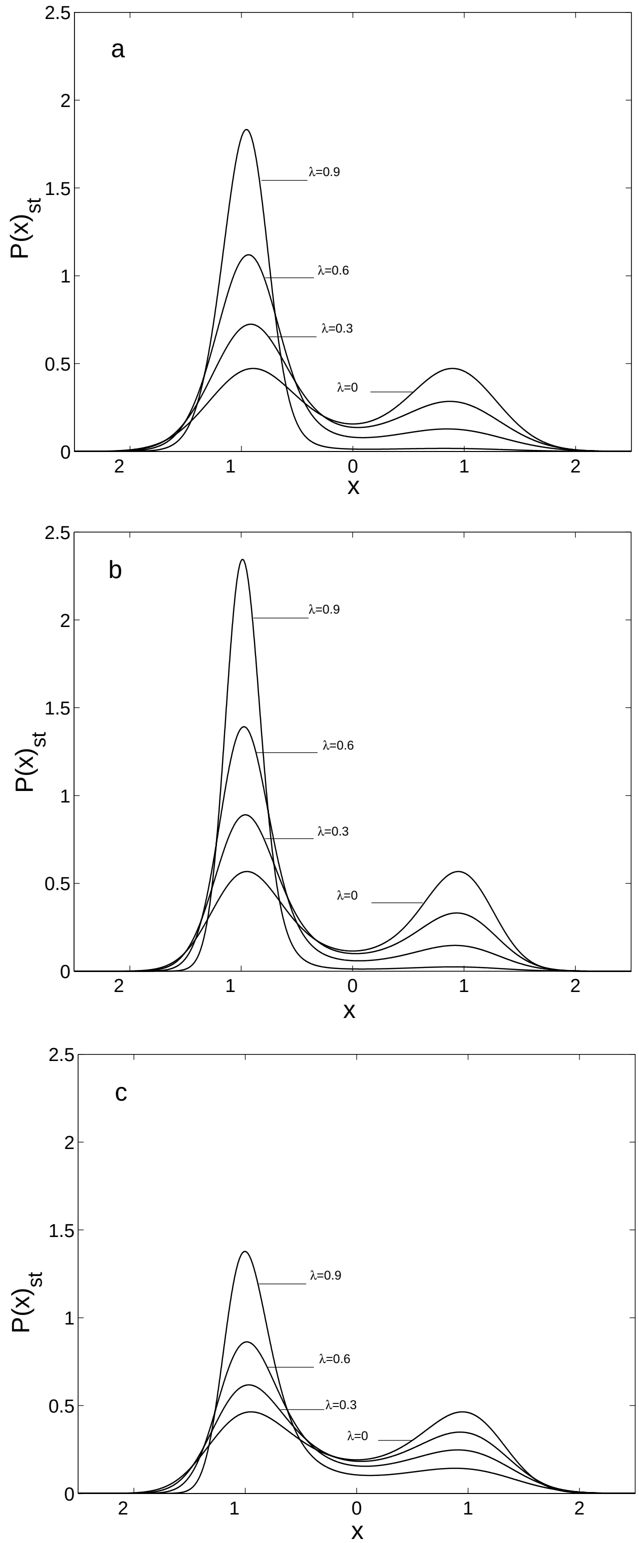
$$\beta_1 = \frac{\lambda}{(1+2\tau)\sqrt{R}}\left[1 - \frac{\left[\left(\frac{2\lambda}{1+2\tau}\right)^2 - 3\right]}{R}\right],$$

$$\beta_2 = \frac{1}{2R}\left[\frac{\left[\left(\frac{2\lambda}{1+2\tau}\right)^2 - 1\right]}{R} - 1\right]. \tag{12}$$

$N$ in Eq. (10) is the normalization constant and $R=D/\alpha$ is the ratio of noise intensities. It should be pointed out that the correlation time $\tau$ must be zero when the strength $\lambda$ of the correlation between noises is zero, however Eq. (10) is valid when $\tau=0$. Then the expectation values of the $n$th power of the state variable $x$ are given by

$$\langle x^n\rangle_{\mathrm{st}} = \frac{\int_{-\infty}^{+\infty} x^n P_{\mathrm{st}}(x)dx}{\int_{-\infty}^{+\infty} P_{\mathrm{st}}(x)dx}. \tag{13}$$

According to the expression SPD [Eq. (10)] of the system, the effects of both $\lambda$ and $\tau$ on SPD can be studied by the numerical computation. The SPDs as functions of $\lambda$ and $\tau$ are plotted in Figs. 1(a)–2(c). Figures 1(a) and 1(b) show that the larger $\lambda$ is, the sharper one peak of the SPD is along with another peak disappearing for cases of $R\geqslant 1$. For the case of $R<1$, the effect of $\lambda$ on the SPD is weaker [see Fig. 1(c)]. That is, for the cases of $R\geqslant 1$, the larger $\lambda$ is, the smaller fluctuation of the state variable $x$ is. From Figs. 2(a) and 2(b), one can also see that the smaller $\tau$ is, the sharper one peak of the SPD is along with another peak disappearing. For the case of $R<1$, the effect of $\tau$ on the SPD is weaker [see Fig. 2(c)]. That is, for the cases of $R\geqslant 1$, the larger $\tau$ is, the larger fluctuation of the state variable $x$ is. Therefore, the mean of the stationary state variable $\langle x\rangle_{\mathrm{st}}$ decreases with increasing $\lambda$ and increases with increasing $\tau$. The effects of both $\lambda$ and $\tau$ on $\langle x\rangle_{\mathrm{st}}$ for the cases of $R\geqslant 1$ are more pronounced than that for the case of $R<1$.



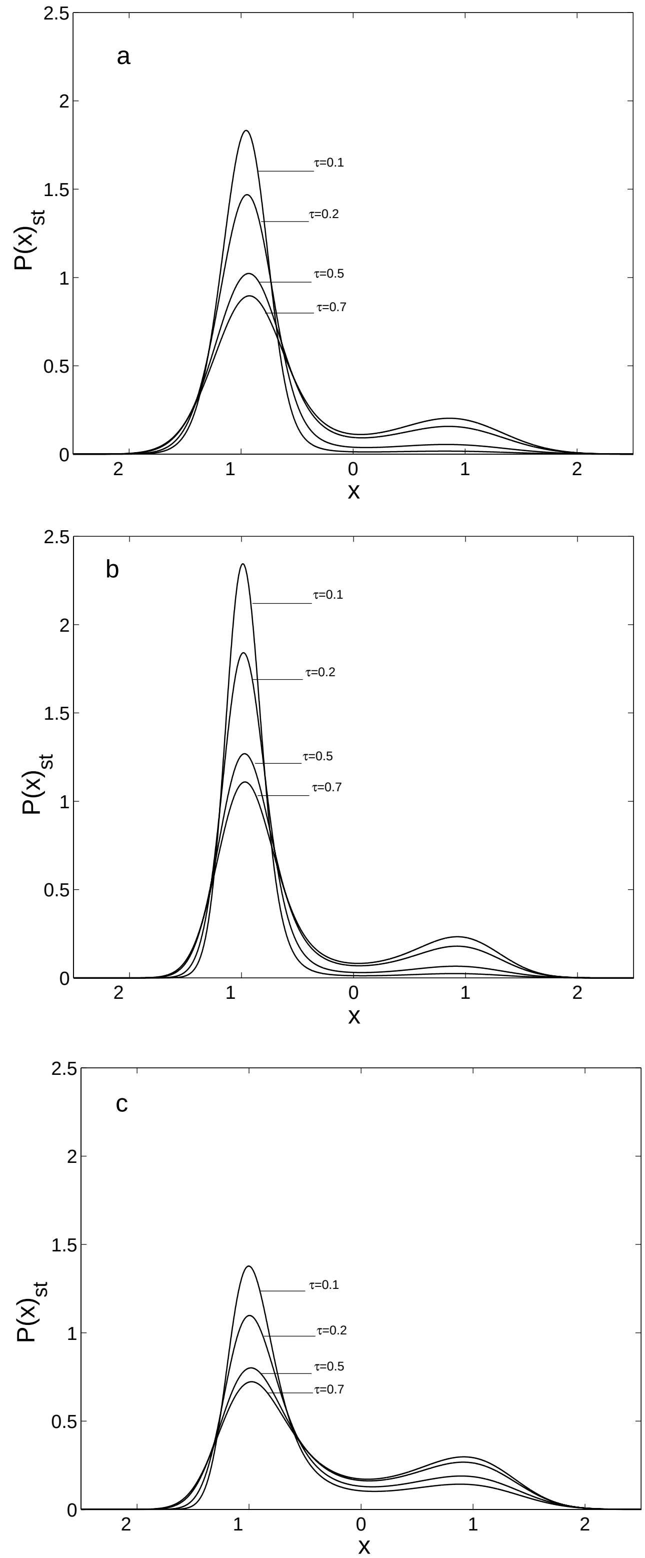




FIG. 1. The stationary probability distribution $P(x)_{st}$. (a) $D=0.2$, $\alpha=0.1$ (i.e., $R>1$), and $\tau=0.1$ are fixed. λ takes 0.9, 0.6, 0.3, and 0. (b) $D=0.1$, $\alpha=0.1$ (i.e., $R=1$), and $\tau=0.1$ are fixed. λ takes 0.9, 0.6, 0.3, and 0. (c) $D=0.1$, $\alpha=0.2$ (i.e., $R<1$), and $\tau=0.1$ are fixed. λ takes 0.9, 0.6, 0.3, and 0.

FIG. 2. The stationary probability distribution $P(x)_{st}$. (a) $D=0.2$,$\alpha=0.1$ (i.e., $R>1$), and $\lambda=0.9$ are fixed. $\tau$ takes 0.7, 0.5, 0.2, and 0.1. (b) $D=0.1$, $\alpha=0.1$ (i.e., $R=1$), and $\lambda=0.9$ are fixed. $\tau$ takes 0.7, 0.5, 0.2, and 0.1. (c) $D=0.1$, $\alpha=0.2$ (i.e., $R<1$), and $\lambda=0.9$ are fixed. $\tau$ takes 0.7, 0.5, 0.2, and 0.1.

### B. Relaxation time of the bistable system

The stationary normalized correlation function of the state variable $x$ is defined by

$$C(s)=\frac{\langle \delta x(t+s)\delta x(t)\rangle_{\rm st}}{\langle(\delta x)^2\rangle_{\rm st}}. \tag{14}$$

It describes the fluctuation decay of a dynamical variable $\delta x=x-\langle x\rangle_{\rm st}$ in the stationary state. In terms of the adjoint operator $L_{\rm FP}^{+}$ of Eq. (7), $\delta x(t+s)$ can be expressed as

$$\delta x(t+s)=\exp(L_{\rm FP}^{+}s)\,\delta x(t). \tag{15}$$

Thus one can rewrite Eq. (14), and get the associated Laplace transform

$$\begin{aligned}\tilde{C}(\omega)&=\int_0^\infty ds\,\exp(-\omega s)C(s)\\&=\frac{1}{\langle(\delta x)^2\rangle_{\rm st}}\left\langle \delta x\frac{1}{\omega-L_{\rm FP}^{+}}\delta x\right\rangle_{\rm st}.\end{aligned} \tag{16}$$

The fluctuation decay of the dynamical variable also can be represented by the relaxation time $T_c$, and it is defined by

$$T_c=\int_0^{+\infty}C(t)dt \tag{17}$$

and

$$T_c=\tilde{C}(0). \tag{18}$$

From the correlation function, one can derive the relaxation time. In order to deal with the Laplace resolvent $(s-L^{+})$ in Eq. (16), by virtue of the projection-operator method used by Fujisaka and Grossmann [19], one has the following continued fraction expression:

$$\tilde{C}(\omega)=\cfrac{1}{\omega+\gamma_0+\cfrac{\eta_1}{\omega+\gamma_1+\cfrac{\eta_2}{\omega+\gamma_2+\cdots}}}, \tag{19}$$

where

$$\gamma_i=-\frac{\langle\delta x_i L^{+}\delta x_i\rangle_{\rm st}}{\langle(\delta x_i)^2\rangle_{\rm st}}, \tag{20a}$$

$$\eta_i=-\frac{\langle(\delta x_i)^2\rangle_{\rm st}}{\langle(\delta x_{i-1})^2\rangle_{\rm st}}, \tag{20b}$$

$$\delta x_{i+1}=Q_{i+1}L^{+}\delta x_i, \tag{20c}$$

starting with $\delta x_0=\delta x$ and $Q_0=1$. The operator $Q_i$ is determined by

$$P_{i-1}=Q_{i-1}-Q_i=\frac{\delta x_{i-1}}{\langle(\delta x_{i-1})^2\rangle_{\rm st}}(\delta x_{i-1}|, \tag{20d}$$

where the operator $(\delta x_i|$ acting on an arbitrary function of state variable, $\varphi(x)$, means the scalar product

$$(\delta x_i|\varphi(x)=\langle(\delta x_i\varphi(x))\rangle_{st}=\int dx P_{\rm st}(x)\,\delta x_i\varphi(x). \tag{20e}$$

In other words, the projection operator $P_i$ projects $\varphi(x)$ onto the subspace associated with the variable $\delta x_i$. The projector $Q_i$ projects onto the space orthogonal to the space containing $\delta x_i$. The basic idea behind the method used to lead a continued fraction expansion (19) is to identify $\delta x_i$ as a slow variable in $Q_i$ space and it slaves the remaining fast variables [15].

After manipulating the zeroth-order approximation of the relaxation time $T_c$, by setting $\eta_1=0$, it is given by

$$T_c=\gamma_0^{-1}=\frac{\langle(\delta x)^2\rangle_{\rm st}}{\langle G(x)\rangle_{\rm st}}. \tag{21}$$

The zeroth approximation (21) of $T_c$ obtained by truncating Eq. (19) is in agreement with that by virtue of the Stratonovich-like ansatz [20]. Obviously, the zeroth approximation (21) is the relaxation time when the memory kernel

$$K(\omega)=-\cfrac{\eta_1}{\omega+\gamma_1+\cfrac{\eta_2}{\omega+\gamma_2+\cdots}} \tag{22}$$

is completely neglected. Fujisaka and Grossmann [19] pointed out that earlier experience with the Duffing oscillator or with the laser fluctuations has shown that the effects of higher orders of memory are not significant. Setting $\eta_2=0$, the first-order approximation of $T_c$ reads

$$T_c=\left[\gamma_0+\frac{\eta_1}{\gamma_1}\right]^{-1}, \tag{23}$$

where

$$\eta_1=\frac{\langle G(x)f'(x)\rangle_{\rm st}}{\langle(\delta x)^2\rangle_{\rm st}}+\gamma_0^2,$$

$$\gamma_1=-\frac{\langle G(x)[f'(x)]^2\rangle_{\rm st}}{\eta_1\langle(\delta x)^2\rangle_{\rm st}}+\frac{\gamma_0^3}{\eta_1}-2\gamma_0. \tag{24}$$

Using Eqs. (8)–(13) and (24), we have

$$\gamma_0=\frac{D\langle x^2\rangle_{\rm st}+\alpha}{\langle(\delta x)^2\rangle_{\rm st}}+\frac{2\lambda\sqrt{D\alpha}\langle x\rangle_{\rm st}}{(1+2\tau)\langle(\delta x)^2\rangle_{\rm st}},$$

$$\eta_1=\gamma_0(\gamma_0-5D-2)-\frac{3\kappa_1}{\langle(\delta x)^2\rangle_{\rm st}},$$

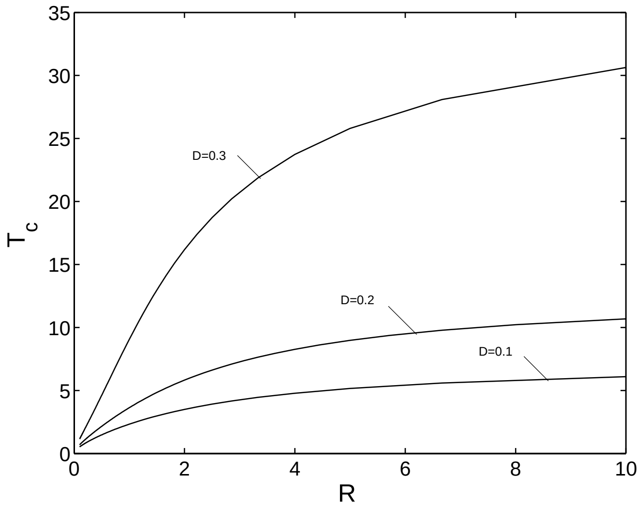



FIG. 3. The relaxation time $T$ vs $R$ for $\lambda=0$ and $\tau=0$. $D$ takes 0.3 , 0.2, and 0.1.

$$\gamma_1 = \frac{\gamma_0^3 + 2(1+D)\gamma_0^2 - \gamma_0\kappa_2}{\eta_1} - \frac{9\kappa_3}{\eta_1\langle(\delta x)^2\rangle_{st}} - 2\gamma_0 - 2(1+D), \quad (25)$$

where

$$\kappa_1 = \alpha\langle x^2\rangle_{st} + \frac{D\lambda\sqrt{D\alpha}}{1+2\tau}\langle x\rangle_{st} + \alpha\left[2D\left(\frac{\lambda}{1+2\tau}\right)^2 - D - 1\right],$$

$$\kappa_2 = 71D^2 + 52D + 45\alpha + 8,$$

$$\kappa_3 = 10D\alpha\left(\frac{\lambda}{1+2\tau}\right)^2\langle x^2\rangle_{st} + \frac{\lambda}{1+2\tau}\sqrt{D\alpha}(9D^2 - 3\alpha + 5D) \times\langle x\rangle_{st} + D^2\alpha\left[13\left(\frac{\lambda}{1+2\tau}\right)^2 - 4\right] - 4\alpha(D+\alpha) + \kappa_1. \quad (26)$$

Above results fall back to Eqs. (2.29)–(2.31) presented in Ref. [15] by taking $\lambda=0$ and $\alpha=0$.

Making use of the expressions [Eqs. (23)–(26)] of the relaxation time, the effects of both $\lambda$ and $\tau$ on the relaxation time of the system can be analyzed by the numerical calculation. The results of the numerical calculation of the relaxation time $T_c$ as a function of $R$, $\lambda$, and $\tau$ are plotted in Figs. 3–7, respectively. All quantities plotted are dimensionless as those in Ref. [12].

When there is correlation between the additive and the multiplicative noise (i.e., $\lambda=0$) in the system, $T_c$ increases monotonously with $R$, which is shown in Fig. 3. The multiplicative noise always slows down the fluctuation decay of dynamical variable $x$ when there is no correlation between the additive and the multiplicative noise. This behavior is in agreement with that of Hernandez *et al.* [15] and the simulation result in Ref. [14].

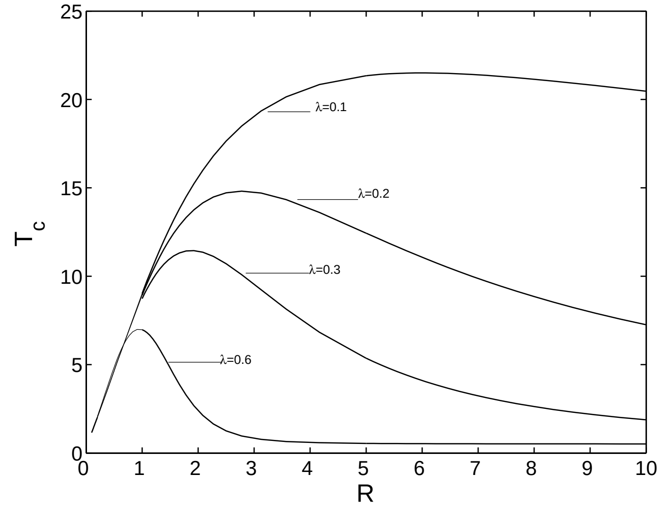



FIG. 4. The relaxation time $T$ vs $R$ for $D=0.1$ and $\tau=0$. $\lambda$ takes 0.1, 0.2, 0.3, and 0.6.

When there are correlations between the additive and the multiplicative noise (i.e., $\lambda\neq 0$) in the system, $T_c$ as a function of $R$ is not monotonic. $T_c$ increases as $R$ increases for smaller values of $R$, however for larger values of $R$, $T_c$ decreases as $R$ increases. There is one peak structure on the $T_c$-$R$ plot (see Fig. 4). The peak position of $T_c$ moves to smaller values of $R$ when values of $\lambda$ increase and $T_c$ decreases as $\lambda$ increases. For the case of smaller $R$, behavior of $T_c$ is the same as that in the case of $\lambda=0$. For a case of larger $R$, $\lambda$ plays an important role in $T_c$.

$T_c$ as a function of $R$ for different values of $\tau$ (i.e., $\tau=0,0.1,0.2,0.3$) are plotted in Fig. 5, which also shows that $T_c$ increases as $R$ increases for smaller values of $R$, but for larger values of $R$, $T_c$ decreases as $R$ increases. There is also a one-peak structure on the $T_c$-$R$ plot. However the peak position of $T_c$ moves to larger values of $R$ when values of $\tau$ increase and $T_c$ increases as $\tau$ increases.

$T_c$ as a function of $\lambda$ for the three cases of $R$ (i.e., $R>1$, $R=1$, and $R<1$) are plotted in Fig. 6, which clearly

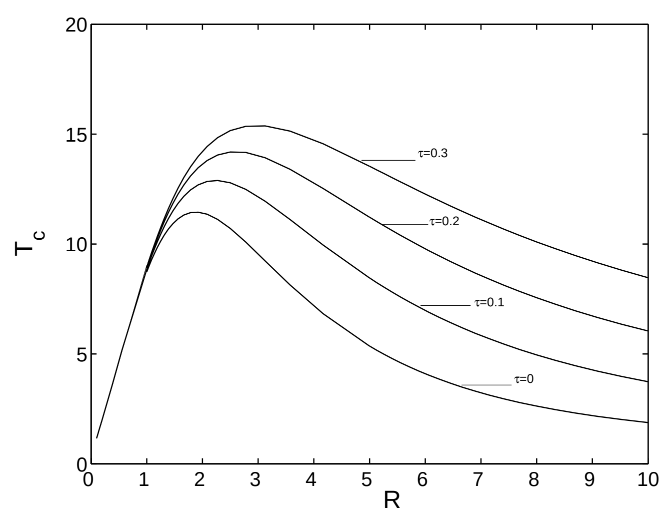



FIG. 5. The relaxation time $T$ vs $R$ for $D=0.1$ and $\lambda=0.3$. $\tau$ takes 0.3, 0.2, 0.1, and 0.

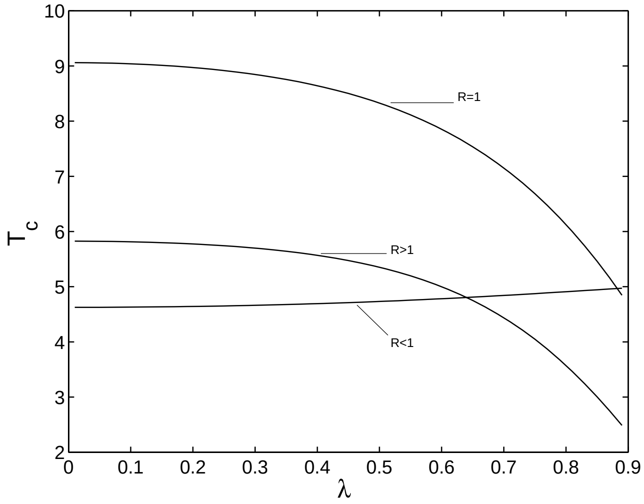



FIG. 6. The relaxation time $T$ vs λ for $\tau$=0.1. $R=1$ ($D=\alpha=0.1$), $R<1$ ($D=0.1,\alpha=0.2$), and $R>1$ ($D=0.2,\alpha=0.1$).

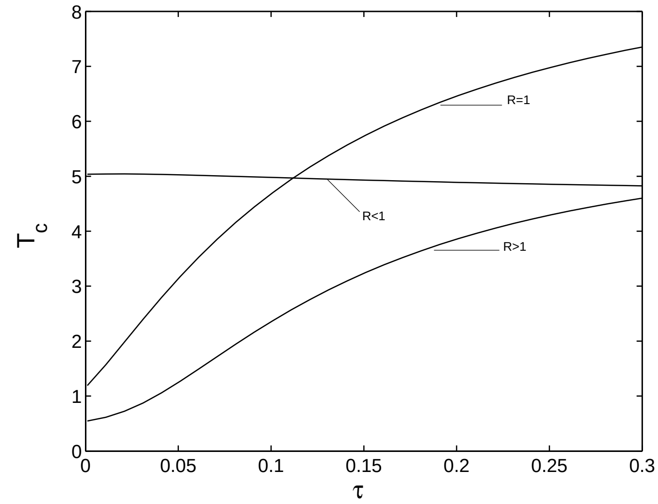



FIG. 7. The relaxation time $T$ vs $\tau$ for λ=0.9. $R=1$ ($D=\alpha=0.1$), $R<1$ ($D=0.1,\alpha=0.2$), and $R>1$ ($D=0.2,\alpha=0.1$).

shows that $T_c$ decreases as λ increases for the cases of $R\geqslant 1$, however for the case of $R<1$, $T_c$ almost does not change as λ increases.

$T_c$ as a function of $\tau$ for the three cases of $R$ (i.e., $R>1, R=1$, and $R<1$) are plotted in Fig. 7. From Fig. 7, one can also clearly see that $T_c$ increases as $\tau$ increases for the cases of $R\geqslant 1$, however for the case of $R<1$, $T_c$ almost does not change as $\tau$ increases.

Because effects of both λ and $\tau$ on $\langle x\rangle_{\rm st}$ for the cases of $R\geqslant 1$ are more pronounced than that for the case of $R<1$, the effects of both λ and $\tau$ on $T_c$ for the cases of $R\geqslant 1$ are more pronounced than that for the case of $R<1$. The behavior of $T_c$ changing with both λ and $\tau$ are different for the cases of $R\geqslant 1$ and the case of $R<1$.

### III. CONCLUSIONS

In this paper, we have studied the effects of both the correlation intensity λ and the correlation time $\tau$ between additive and multiplicative noise on the relaxation time of a conventional bistable system driven by cross-correlated noise. First, we discussed effects of both λ and $\tau$ on the mean of the stationary state variable $\langle x\rangle_{\rm st}$ of the system for three cases of $R$. The effects of both λ and $\tau$ on $\langle x\rangle_{\rm st}$ for the cases of $R\geqslant 1$ are more pronounced than that for the case of $R<1$. Second, the analytic expression for the relaxation time of the system is derived by the projection-operator method. Making use of the expression of the relaxation time, the effects of both λ and $\tau$ on the relaxation time of the system have been analyzed for three cases of $R$. The behavior of $T_c$ for the case of $R<1$ and for the cases of $R\geqslant 1$ are very different. For the case of $R<1$, $T_c$ increases as $R$ increases and for the cases of $R\geqslant 1$, there is one peak structure on the $T_c$-$R$ plot. The effects of both λ and $\tau$ on $T_c$ for the cases of $R\geqslant 1$ are more pronounced than that for the case of $R<1$. For the case of $R<1$, $T_c$ almost does not change with both λ and $\tau$, and for the cases of $R\geqslant 1$, $T_c$ decreases as λ increases, however $T_c$ increases as $\tau$ increases. λ and $\tau$ play opposite roles in $T_c$, i.e., λ enhances the fluctuation decay of dynamical variable and $\tau$ slows down the fluctuation decay of the dynamical variable.

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